THE NONADDITIVITY OF THE GENUS

BY

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ABSTRACT

A class of cubic graphs is introduced for which the genus is a nonadditive function of the genus of subgraphs. This provides a small (28 node) counterexample to Duke's conjecture concerning the relation of the Betti number to the genus of a graph.

Kagno [3] (th. 3, p. 55) first showed that the genus of an irreducible graph is an additive function of the genus of each block comprising the graph. Twenty-six years later, Battle et al. [1] made the observation that Kagno's theorem is also true for graphs which are not irreducible. Milgram [4] (p. 14) introduced a class of cubic graphs (irreducible graphs with extra θ) for which, in the orientable case, the genus is a nonadditive function of the genus of subgraphs. In Milgram [5] the central importance of nonoutside cubic graphs for producing a complete list of six cubic graphs which are irreducibly nonrepresentable on the projective plane is emphasized.

It is the intention of this note to emphasize the importance of the nonoutside graphs (and the corresponding nonadditivity theorem) for the computation of the genus. As an application, we compute the genus of a 28 node cubic graph. Since the genus of the graph is four, it provides a counterexample to a conjecture of Duke [2] (p. 817).

The terms and notations follow that of Milgram [4]. The graph H_3 is illustrated in Fig. 1. It is a planar graph but its three nodes of degree one (the free nodes) cannot all be represented in the same 2–cell when H_3 is given a planar representation. H_3 is thus a nonoutside graph.

The *genus* of a graph G, $\gamma(G)$, is the number of handles of the smallest orientable surface on which G has a 2–cell embedding. We are here concerned

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with a method for calculating the genus of a graph in terms of the genus of a subgraph. Let A be a cubic graph and let T_n be a set of n triplets of distinct points of A. Let $H(A,T_n)$ be the cubic graph formed from A and n copies of H_3 by identifying the free nodes of H_3 with each of the three points of a triple of T_n. Let $S(A,T_n)$ be the set of 3^n cubic graphs with $\|A^\circ\|+2n$ nodes; each formed from A by adding a single edge between two points of a triple of T_n.

Let

$$\gamma_{T_n}(A)=\min\gamma(A') \qquad A'\in S(A,T_n)$$

We will obtain $\gamma(H(A,T_n))$ in terms of $\gamma_{T_n}(A)$. For convenience, we will use the labeling of H_3 in Fig. 1.

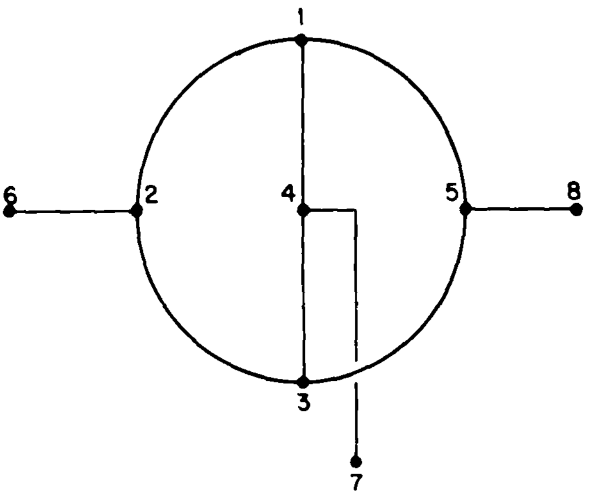

Fig. 1

The *regional number*, $d(A)$ of a graph A is the number of 2–cells into which A divides $S_{\gamma(A)}$.

LEMMA 1. *In any 2-cell embedding of $H(A,T_n)$ in an orientable surface, at most one 2–cell has only boundary nodes* 1, 2, 3, 4, 5 *of a single copy of H_3.*

PROOF. Since nodes 2, 4 and 5 are adjacent to a free node, each of them can only appear (once) on the boundary of one such 2–cell. Thus if there were at least two such 2–cells at least one of the nodes 1 and 3 must appear in both 2–cells. But this means that one of 2, 4 or 5 must appear in both 2–cells which we have shown to be impossible.

LEMMA 2. *Let A be a cubic graph, then:*

$$\gamma(A) < \gamma(H(A, T_1)) < \gamma_{T_1}(A) + 2.$$

PROOF. Since H_3 can not be represented in a 2-cell with all its free nodes in the same 2-cell, $\gamma(H(A, T_1)) > \gamma(A)$. Let A' be one of $S(A, T_1)$ such that $\gamma(A') = \gamma_{T_1}(A)$ and let the added adjacency of A' be (6, 7). Let H be the non-cubic graph formed by identifying (6, 7, 8) with T_1 of A'. Then, adding H_3 on the cylinder of Fig. 2 gives an embedding of H on $S_{\gamma(A')+1}$. Since $H \subset H(A, T_1)$,

$$\gamma(H(A, T_1)) \leqq \gamma(H) = \gamma(A') + 1.$$

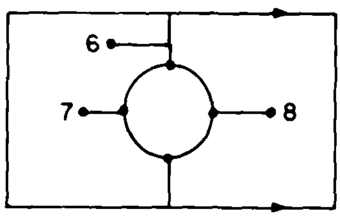

Fig. 2

THEOREM 3. *Let A be a cubic graph, then*

$$\gamma(H(A, T_n)) = \gamma_{T_n}(A) + n.$$

PROOF. By induction on n (the number of copies of H_3). Let $n = 1$. Let A' be one of the three graphs. We note first that $\gamma(H(A, T_1)) \leqq \gamma_{T_1}(A) + 1$ by the previous Lemma. Now, if $\gamma(A') = \gamma(A)$, and $\gamma(H(A,T_1)) = \gamma(A') + 1$, then two of the points of T_1 lie on the same 2-cell when A has some embedding in $S_{\gamma(A)}$. In that case, the addition of H_3 embedded in the cylinder of Fig. 2 gives an embedding of $H(A, T_1)$ on $S_{\gamma(A)+1}$, or, for this case, $\gamma(H(A, T_1)) = \gamma_{T_1}(A) + 1 = \gamma(A) + 1$. Now, each A' can have genus at most one more than that of A since $\|A'^0\| = \|A^0\| + 2$. Assume, then, that each $\gamma(A') = \gamma(A) + 1$. We show in this case, $(H(A, T_1)) \neq \gamma(A) + 1$. Assume the contrary, that $\gamma(H(A, T_1)) = \gamma(A) + 1$. Then by the Euler-Poincaré formula for cubic graphs G,

$$d(G) = \frac{\|G^0\|}{2} + 2 - 2\gamma(G)$$

or

$$d(H(A,T_1)) = \frac{\|(H(A,T_1))^0\|}{2} + 2 - 2\gamma(H(A,T_1))$$

$$= \frac{\|A^0\|}{2} + 2 - 2\gamma(A) + 2$$

$$= d(A) + 2$$

since $\|(H(A,T_1))^0\| = \|A^0\| + 8$. Or, an embedding of $H(A,T_1)$ on $S_{\gamma(A)+1}$ has two more 2–cells than that of the embedding of A in $S_{\gamma(A)}$. But the three points of T_1 are on separate 2–cells of A embedded in $S_{\gamma(A)}$ (by assumption). Thus three cases apply for an embedding of $H(A,T_1)$ in $S_{\gamma(A)+1}$. These are illustrated as (i), (ii) and (iii) in Fig. 3.

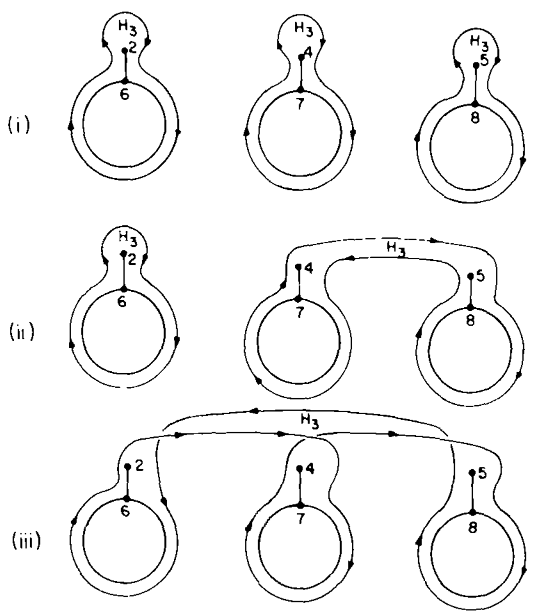

Fig. 3

In this figure, the circles represent the three distinct 2–cells on which the points of A (identified with nodes 6, 7 and 8) lie. (The reader should bear in mind the circles described represent the boundary of a 2–cell only when the vertex-ordering of edges for the embedding of $H(A,T_1)$ is restricted to the graph A.) In the figure, faces are traced by the method of Ringel-Edmonds. In each of the three cases, we are adding structure to a representation of A on $S_{\gamma(A)}$ in order to see how the number of 2–cells will change. The argument above shows that we must end with an additional two 2–cells.

In case (i), the two extra 2–cells must be made only of nodes 1, 2, 3, 4, 5 of H_3 which is impossible by Lemma 1. In case (ii), two 2–cells of A are coalesced and three 2–cells must be made of nodes 1 2 3 4 5 (and the edges between them) which is also impossible. In case (iii), three 2–cells of A are coallesced and four 2–cells must be made of nodes 1 2 3 4 5 which completes the contradiction showing that:

$$\gamma(H(A,T_1)) \geqq \gamma(A)+2 = \gamma_{T_1}(A)+1.$$

Using lemma 2, we conclude for the case where $\gamma_{T_1}(A) = \gamma(A)+1$ that $\gamma(H(A,T_1)) = \gamma_{T_1}(A)+1$, which completes the proof for $n=1$. But, $H(A,T_n) = H(H(A,T_{n-1}),T_N)$ where T_N is the single last triple of the set T_n. Our induction is now complete, and we have proved the theorem.

Remark. It may be though that the role of H_3 in the theorem above could be replaced by the graph Q which is simply a single node with three free edges. The argument would procede by observing that when A is represented on $S_{\gamma(A)}$, the graph $A+Q$ would require at least $\gamma(A)+2$ handles since each of the points of T_1 is in a different 2–cell of $S_{\gamma(A^*)}$. This argument overlooks the fact that A might nevertheless have a representation on $S_{\gamma(A)+1}$ in which all points of T_1 are in the same 2–cell. Thus the genus of $A+Q$ may be $\gamma(A)+1$. The possibility of representing A on $S_{\gamma(A)+1}$ however, insures that $A+H_3$ will be represented on $S_{\gamma(A)+2}$ since H_3 cannot be represented in a 2–cell with all its free edges in the same region.

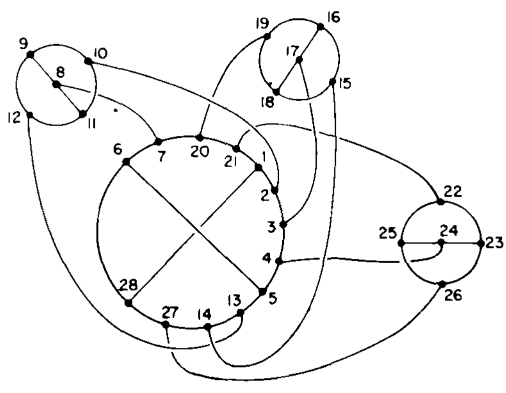

Fig. 4

As an example, we will compute the genus of the 28 node cubic graph G of Fig. 4. $\gamma(G) = 3+\gamma(A')$ where A' is the one of the 27 graphs of Fig. 5 (with three added edges of the form $(a_{jk}, a_{j'k})$, $k = 1,2,3$) which has smallest genus.

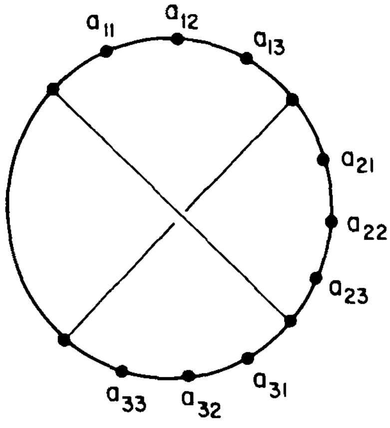

Fig. 5

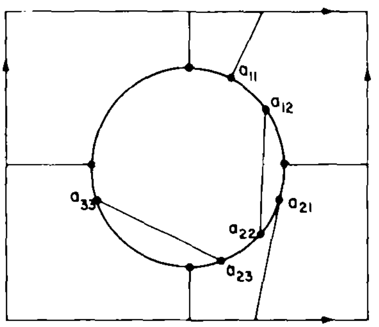

Fig. 6

The three triples for this set of graphs are (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}) and (a_{13}, a_{23}, a_{33}). Since one of the graphs (in Fig. 6) is of genus 1, $\gamma(G) \leqq 4$. All we need to show that $\gamma(G) = 4$ is:

LEMMA 4. *If $A' \in S(A, T_3)$ (defined above) then $\gamma(A') \geqq 1$.*

PROOF. Clearly, any adjacency of the form (a_{1k}, a_{3k}) forms a $K_{3,3}$. There are 19 such cases in $S(A, T_3)$. The remaining 8 cases are characterized by an a_{2k} as one of the two selected points of a triple. Thus, there are two adjacencies of the form (a_{jk}, a_{2k}) and (a_{jm}, a_{2m}) say, (a_{31}, a_{21}) and (a_{33}, a_{23}). Such graphs (Fig. 7) are clearly non-planar, which completes the proof.

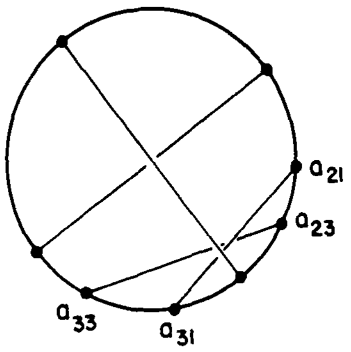

Fig. 7

References

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